

ARRANGEMENTS OF SPHERES AND PROJECTIVE SPACES

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ABSTRACT. We develop the theory of arrangements of spheres. We consider a finite collection codimension 1 spheres in a given finite dimensional sphere. To such a collection we associate two posets: the poset of faces and the poset of intersections. We also associate a topological space to this collection. The complement of union of tangent bundles of these sub-spheres inside the tangent bundle of the ambient sphere which we call the tangent bundle complement. We find a closed form formula for the homotopy type of this complement and express some of its topological invariants in terms of the associated combinatorial information.

INTRODUCTION

An arrangement of hyperplanes is a finite set \mathcal{A} consisting of codimension 1 subspaces of \mathbb{R}^l . These hyperplanes and their intersections induce a stratification of \mathbb{R}^l . The strata (or faces) form a poset (face poset) when ordered by inclusion and the set of all possible intersections forms a poset ordered by reverse inclusion. These posets contain important combinatorial information about the arrangement. An important topological object associated with an arrangement \mathcal{A} is the *complexified complement* $M(\mathcal{A})$. It is the complement of the union of the complexified hyperplanes in \mathbb{C}^l . One of the aspects of the theory of arrangements is to understand the interaction between the combinatorial data of an arrangement and the topology of this complement. For example, the cohomology ring of the complement is completely determined by the intersection data. A pioneering result by Salvetti [4] states that the homotopy type of the complement is determined by the face poset.

A generalization of hyperplane arrangements was introduced by the author in [1]. Where a study of arrangements of codimension 1 submanifolds in a smooth manifold was initiated. In this paper we focus on a particular example of sphere arrangements. Apart from applying general theorems we prove some new results which are specific to arrangements of spheres.

One of the motivations to study hyperplane arrangements comes from its natural connection with reflection groups and associated Artin groups. To every finite reflection group there corresponds an arrangement of hyperplanes that are fixed by the hyperplanes. There is a fixed point free action of the reflection group on the complexified complement. The fundamental group of the corresponding quotient space is the Artin group associated with the (Coxeter presentation of the) reflection group. Topological properties of the complement offers insight into algebraic properties of the Artin group. We investigate sphere arrangements with a similar motivation. Finite reflection groups (or Coxeter groups) are discrete subgroups of isometries of a sphere of appropriate dimension. The fixed point set of the action of reflection groups on a sphere is an arrangement of subspheres of codimension 1. To such an arrangement we associate a topological space called as the tangent bundle complement on which the reflection

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group acts fixed point freely. The fundamental groups of the complement and the quotient space serve as the analogue of pure Artin groups and Artin groups respectively. Main aim of this paper is to lay topological foundations to study these kinds of "Artin like" groups. The results regarding these groups will appear elsewhere.

The paper is organized as follows. Section 1 is about the preliminaries from hyperplane arrangements. In Section 2 we introduce the new objects of study, arrangements of spheres and the tangent bundle complement. In Section 3 we look at how does the combinatorics determines the topology of the complement. In particular we prove a closed form formula for its homotopy type. We look at the fundamental group in Section 4. Finally in Section 5 we look at arrangements of projective spaces.

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1. ARRANGEMENTS OF HYPERPLANE

Hyperplane arrangements arise naturally in geometric, algebraic and combinatorial instances. They occur in various settings such as finite dimensional projective or affine (vector) spaces defined over field of any characteristic. In this section we will formally define hyperplane arrangements and the combinatorial data associated with it in a setting that is most relevant to our work.

Definition 1.1. A real, central *arrangement of hyperplanes* is a collection $\mathcal{A} = \{H_1, \dots, H_k\}$ of finitely many codimension 1 subspaces (*hyperplanes*) in \mathbb{R}^l , $l \geq 1$. Here l is called as the *rank* of the arrangement.

If we allow \mathcal{A} to contain affine hyperplanes (i.e., translates of codimension 1 subspaces) we call \mathcal{A} an affine arrangement. However we will mostly consider central arrangements. Hence, an arrangement will always mean central, unless otherwise stated. We also assume that all our arrangements are *essential*, it means that the intersection of all the hyperplanes is the origin. For an affine subspace X of \mathbb{R}^l , the *contraction* of X in \mathcal{A} is given by the subarrangement $\mathcal{A}_X := \{H \in \mathcal{A} \mid X \subseteq H\}$. The hyperplanes of \mathcal{A} induce a stratification (cellular decomposition) on \mathbb{R}^l , components of each stratum are called *faces*.

There are two posets associated with \mathcal{A} , namely, the face poset and the intersection lattice which contain important combinatorial information about the arrangement.

Definition 1.2. The *intersection lattice* $L(\mathcal{A})$ of \mathcal{A} is defined as the set of all intersections of hyperplanes ordered by reverse inclusion.

$$L(\mathcal{A}) := \{X := \bigcap_{H \in \mathcal{B}} H \mid \mathcal{B} \subseteq \mathcal{A}, X \neq \emptyset\}, \quad X \geq Y \Leftrightarrow X \subseteq Y$$

Note that for affine arrangements, set of all intersections only form a poset and not a lattice.

Definition 1.3. The *face poset* $\mathcal{F}(\mathcal{A})$ of \mathcal{A} is the set of all faces ordered by inclusion: $F \leq G$ if and only if $F \subseteq \overline{G}$.

Codimension 0 faces are called *chambers*, the set of all chambers will be denoted by $\mathcal{C}(\mathcal{A})$. A chamber is bounded if and only if it is a bounded subset of \mathbb{R}^l . Two chambers C and D are adjacent if they have a common face.

An interesting space associated with a real hyperplane arrangement \mathcal{A} is its *complexified complement* $M(\mathcal{A})$ which is defined as follows:

Definition 1.4.

$$M(\mathcal{A}) := \mathbb{C}^l \setminus \left(\bigcup_{H \in \mathcal{A}} H_{\mathbb{C}} \right)$$

where $H_{\mathbb{C}}$ is the hyperplane in \mathbb{C}^l with the same defining equation as $H \in \mathcal{A}$.

1.1. The Salvetti Complex. There is a construction of a regular CW-complex, introduced by Salvetti [4], which models the homotopy type of the complexified complement. Let us first describe its cells and how they are attached. The k -cells of $Sal(\mathcal{A})$ are in one-to-one correspondence with the pairs $[F, C]$, where F is a codimension k face of the given arrangement and C is a chamber whose closure contains F . A cell labeled $[F_1, C_1]$ is contained in the boundary of another cell $[F_2, C_2]$ if and only if $F_1 \leq F_2$ in $\mathcal{F}(\mathcal{A})$ and C_1, C_2 are contained in the same chamber of (the arrangement) \mathcal{A}_{F_1} (with the attaching maps being homeomorphisms).

Theorem 1.5 (Salvetti [4]). *Let \mathcal{A} be an arrangement of real hyperplanes and $M(\mathcal{A})$ be the complement of its complexification inside \mathbb{C}^l . Then there is an embedding of $Sal(\mathcal{A})$ into $M(\mathcal{A})$ moreover there is a natural map in the other direction which is a deformation retraction.*

1.2. Cohomology of the Complement. Let us start by defining the *Orlik-Solomon algebra* associated with an arrangement. The construction of the Orlik-Solomon algebra is completely combinatorial. This algebra is also defined for complex arrangements (where hyperplanes are defined using complex equations).

Let E_1 be the free \mathbb{Z} -module generated by the elements e_H for every $H \in \mathcal{A}$. Define $E(\mathcal{A})$ to be the exterior algebra on E_1 . For $S = (H_1, \dots, H_p)$ ($1 \leq p \leq n$), call S independent if $rank(\cap S) := \dim(H_1 \cap \dots \cap H_p) = p$ and dependent if $rank(\cap S) < p$. Notice the unfortunate clash of notations, this rank is different from the one used in the intersection lattice. Geometrically independence implies that the hyperplanes of S are in general position. Let $I(\mathcal{A})$ denote the ideal of E generated by all $\partial e_S := \partial(e_{H_1} \cdots e_{H_p})$, where S is a dependent tuple and ∂ is the differential in E .

Definition 1.6. The *Orlik-Solomon algebra* of a complexified central arrangement \mathcal{A} is the quotient algebra $A(\mathcal{A}) := E(\mathcal{A})/I(\mathcal{A})$.

The following important theorem shows how cohomology of $M(\mathcal{A})$ depends on the intersection lattice. It combines the work of Arnold, Brieskorn, Orlik and Solomon. For details and exact statements of their individual results see [3, Chapter 3, Section 5.4].

Theorem 1.7. *Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a complex arrangement in \mathbb{C}^l . For every hyperplane $H_i \in \mathcal{A}$ choose a linear form $l_i \in (\mathbb{C}^l)^*$, such that $\ker(l_i) = H_i$ ($1 \leq i \leq n$). Then the integral cohomology algebra of the complement is generated by the classes*

$$\omega_i := \frac{1}{2\pi} \frac{dl_i}{l_i},$$

for $1 \leq i \leq n$. The map $\gamma: A(\mathcal{A}) \rightarrow H^*(M(\mathcal{A}), \mathbb{Z})$ defined by

$$\gamma(e_H) \mapsto \omega_H$$

induces an isomorphism of graded \mathbb{Z} -algebras.

This theorem asserts that a presentation of the cohomology algebra of $M(\mathcal{A})$ can be constructed from the data that are encoded by the intersection lattice. Let us state one more theorem that explicitly states the role of intersection lattice in determining the cohomology of the complement.

Theorem 1.8. *Let \mathcal{A} be a nonempty complex arrangement and for $X \in L(\mathcal{A})$ let $M_X := M(\mathcal{A}_X)$. For $k \geq 0$ there are isomorphisms*

$$\theta_k: \bigoplus_{X \in L_k} H^k(M_X) \rightarrow H^k(M)$$

induced by the maps $i_X: M \hookrightarrow M_X$ (where $L_k \subset L(\mathcal{A})$ consists of elements of rank k).

2. ARRANGEMENTS OF SPHERES

We now study arrangements of codimension 1 sub-spheres in a sphere. In general the codimension 1 sub-spheres in a sphere could be very difficult to deal with. For example, consider the Alexander horned sphere. It is an embedding of S^2 inside S^3 whose complement is not even simply connected. In order to avoid such pathological instances we restrict our selves to a nice class of spheres.

Definition 2.1. Let S^l denote the unit sphere in \mathbb{R}^{l+1} , a subset S of the unit sphere is called a *hypersphere* if and only if it is neither empty nor singleton and $S = H \cap S^l$ for some (affine) hyperplane H in \mathbb{R}^{l+1} .

An important property, that will be relevant to us, is that for S^l ($l \geq 2$) the complement of a hypersphere contains exactly two connected components homeomorphic to an open ball. There is, in fact, a larger class of codimension 1 spheres called the *tame* spheres whose complement is two open balls. Moreover the tame spheres are homeomorphic to hyperspheres. The reason we do not consider the tame spheres is that given a collection of these spheres there need not exist a homeomorphism taking them to a collection of hyperspheres (see [1, Chapter 5] for an example).

Definition 2.2. An *arrangement of spheres* in the unit sphere S^l is a finite collection $\mathcal{A} = \{S_1, \dots, S_k\}$ of hyperspheres satisfying the following conditions:

- (1) $\mathcal{A}_I := \bigcap_{i \in I} S_i$ is a sphere of some dimension, for all $I \subseteq \{1, \dots, k\}$.
- (2) If $\mathcal{A}_I \not\subseteq S_i$, for some I and $i \in \{1, \dots, k\}$, then $\mathcal{A}_I \cap S_i$ is a hypersphere in \mathcal{A}_I .
- (3) The hyperspheres in \mathcal{A} decompose S^l into a regular cell complex.

If all the hyperspheres are obtained by intersecting with the codimension 1 subspaces then we call such an arrangement a *centrally symmetric arrangement of spheres*.

We assume that the empty set is the unit sphere of dimension -1 and that S^0 consists of two points. For $S \in \mathcal{A}$ let H_S denote the hyperplane in \mathbb{R}^{l+1} such that $S = H_S \cap S^l$. Also because of the above definition all the sphere arrangements we consider satisfy the following non-degeneracy condition

$$\dim(\mathcal{A}_I) < \dim(\bigcap_{S \in \mathcal{A}_I} H_S) \text{ for every non-empty subset } I.$$

We should also note here that the sphere arrangements are a special case of submanifold arrangements. These arrangements were introduced in [1, Chapter 3]. Interested reader is invited to check that the sphere arrangements satisfy all the conditions for submanifold arrangements.

Similar to the case of hyperplane arrangements the combinatorial information associated with sphere arrangements is contained in the two posets which we now define.

Definition 2.3. The *intersection poset* denoted by $L(\mathcal{A})$ is the set of connected components of all possible intersections of S_i 's ordered by reverse inclusion, by convention $S^l \in L(\mathcal{A})$ as the smallest element. The rank of each element in $L(\mathcal{A})$ is defined to be the codimension of the corresponding intersection.

Definition 2.4. The intersections of these S_i 's in \mathcal{A} define a stratification of S^l . The connected components in each stratum are called *faces*. Top dimensional faces are called chambers and the set of all chambers is denoted by $\mathcal{C}(\mathcal{A})$. The collection of all the faces $\mathcal{F}(\mathcal{A}) = \cup \mathcal{F}^i(\mathcal{A})$ is the *face poset* with the ordering $F \leq G \Leftrightarrow F \subseteq \overline{G}$. It is a graded poset and the rank of each face is its dimension.

Here are two examples of sphere arrangements in dimensions 1 and 2.

Example 2.5. Let X be the circle S^1 , a smooth one dimensional manifold, the codimension 1 submanifolds are points in S^1 . Consider the arrangement $\mathcal{A} = \{p, q\}$ of 2 points. For both these points there is an open neighborhood which is homeomorphic to an arrangement of a point in \mathbb{R} . Figure 1 shows this arrangement and the Hasse diagrams of the face poset and the intersection poset.

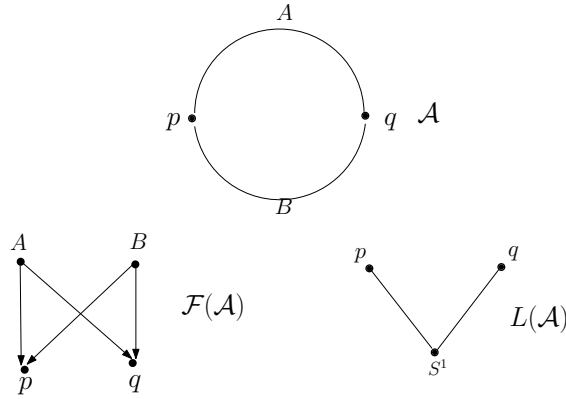


FIGURE 1. Arrangement of 2 points in a circle.

Example 2.6. As a 2-dimensional example consider an arrangement of 2 great circles N_1, N_2 in S^2 . Figure 2 shows this arrangement and the related posets. The face poset has two 0-cells, four 1-cells and four 2-cells. Also note that the order complex of the face poset has the homotopy type of S^2 .

A hypersphere S of \mathcal{A} *separates* two chambers C and D if and only if they are contained in the distinct connected components of $S^l \setminus S$. For two chambers C, D the set of all the hyperspheres that separate these two chambers is denoted by $R(C, D)$. The following lemma is now evident.

Lemma 2.7. Let S^l be the l -sphere and \mathcal{A} be an arrangement of spheres, let C_1, C_2, C_3 be three chambers of this arrangement. Then,

$$R(C_1, C_3) = [R(C_1, C_2) \setminus R(C_2, C_3)] \cup [R(C_2, C_3) \setminus R(C_2, C_1)].$$

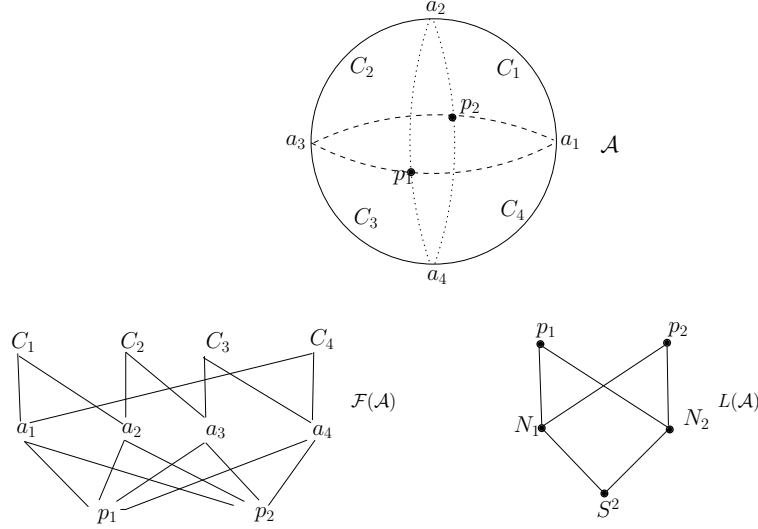


FIGURE 2. Arrangement of 2 circles in a sphere.

The *distance between two chambers* is defined as the cardinality of $R(C, D)$ and denoted by $d(C, D)$. Given a face F and a chamber C of a sphere arrangement \mathcal{A} , define the action of F on C as follows:

Definition 2.8. A face F acts on a chamber C to produce another chamber $F \circ C$ satisfying:

- (1) $F \subset \overline{F \circ C}$
- (2) $d(C, F \circ C) = \min \{d(C, C') \mid C' \in \mathcal{C}(\mathcal{A}), F \subset \overline{C'}\}$.

Lemma 2.9. *With the same notation as above, the chamber $F \circ C$ always exists and is unique.*

Proof. First, note that $F \circ C = F$ if and only if F itself is a chamber. Let $\text{codim} F \geq 1$ so, $F = N_1 \cap \dots \cap N_r$. Let $\mathcal{A}' = \{N_1, \dots, N_r\}$, the collection \mathcal{A}' need not be an arrangement of spheres. However, \mathcal{A}' defines a stratification of the l -sphere and we will refer to the codimension 0 components of this stratification as chambers. There exists a unique chamber C' of \mathcal{A}' which contains C . Then define $F \circ C$ to be the unique chamber of \mathcal{A} that is contained in C' and whose closure contains F . \square

If $F', F' \geq F$ are two faces then $F' \circ (F \circ C) = F' \circ C$. Also, if $F \subseteq \overline{C}$ then $F \circ C = C$.

Associated with such a sphere arrangement is the complement of the union of the hyperspheres which is obviously disconnected. The problem of counting connected components of this complement is studied in [1, Chapter 6]. This number is determined by the intersection poset and generalizes the seminal result due to Zaslavsky for hyperplane arrangements. However, the topological space we are interested in this paper is defined in the following section.

2.1. The tangent bundle complement. If one were to forget the complex structure on \mathbb{C}^l then, topologically, it is the tangent bundle of \mathbb{R}^l . Same is true for a hyperplane H and its complexification $H_{\mathbb{C}}$. Hence the complexified complement of a hyperplane arrangement can also be considered as a complement inside the tangent bundle. We use this topological viewpoint to define a generalization of the complexified complement for sphere arrangements.

Definition 2.10. Let S^l be a l -dimensional sphere and $\mathcal{A} = \{N_1, \dots, N_k\}$ be an arrangement of hyperspheres. Let TS^l denote the tangent bundle of S^l and let $T\mathcal{A} := \bigcup_{i=1}^k TN_i$. The **tangent bundle complement** of the arrangement \mathcal{A} is defined as

$$M(\mathcal{A}) := TS^l \setminus T\mathcal{A}.$$

It is connected as it is of codimension 2 in TS^l .

The above space was introduced in [1, Chapter 3] in the context of submanifold arrangements. We now construct a regular cell complex that has the homotopy type of the tangent bundle complement.

Let \mathcal{A} be an arrangement of spheres in a l -sphere S^l and let $\mathcal{F}(\mathcal{A})$ denote the associated face poset. By $(S^l, \mathcal{F}(\mathcal{A}))$ we mean the regular cell structure of S^l induced by the arrangement. Let $\mathcal{F}^*(\mathcal{A})$ denote the dual face poset and we denote by pair $(S^l, \mathcal{F}^*(\mathcal{A}))$ the *dual cell structure*. Every k -cell in $(S^l, \mathcal{F}(\mathcal{A}))$ corresponds to $(l - k)$ -cell in $(S^l, \mathcal{F}^*(\mathcal{A}))$ for $0 \leq k \leq l$. The partial order on the cells of $\mathcal{F}^*(\mathcal{A})$ will be denoted by \prec .

For the sake of notational simplicity we will denote the dual cell complex by $\mathcal{F}^*(\mathcal{A})$ (and by \mathcal{F}^* if the context is clear). The symbols C, D will denote vertices of \mathcal{F}^* and the symbol F^k will denote a k -cell dual to the codimension k -face F of \mathcal{A} . Note that a 0-cell C is a vertex of a k -cell F^k in \mathcal{F}^* if and only if the closure \bar{C} of the corresponding chamber contains the $(l - k)$ -face F . The action of the faces on chambers that was introduced above is also valid for the dual cells. The symbol $F^k \circ C$ will denote the vertex of F^k which is dual to the unique chamber closest to C .

Given a hypersphere arrangement \mathcal{A} in S^l construct a regular CW complex $Sal(\mathcal{A})$ of dimension l as follows:

The 0-cells of $Sal(\mathcal{A})$ correspond to 0-cells of \mathcal{F}^* , which we denote by the pairs $\langle C; C \rangle$. For each 1-cell $F^1 \in X^*$ with vertices C_1, C_2 , assign two homeomorphic copies of F^1 denoted by $\langle F^1; C_1 \rangle, \langle F^1; C_2 \rangle$. Attach these two 1-cells in $Sal(\mathcal{A})_0$ (the 0-skeleton) such that

$$\partial \langle F^1; C_i \rangle = \{ \langle C_1; C_1 \rangle, \langle C_2; C_2 \rangle \}.$$

Orient the 1-cell $\langle F^1; C_i \rangle$ so that it begins at $\langle C_i; C_i \rangle$, to obtain an oriented 1-skeleton $Sal(\mathcal{A})_1$.

By induction assume that we have constructed the $(k - 1)$ -skeleton of $Sal(\mathcal{A})$, $1 \leq k - 1 < l$. To each k -cell $F^k \in \mathcal{F}^*$ and to each of its vertex C assign a k -cell $\langle F^k; C \rangle$ that is isomorphic to F^k . Let $\phi(F^k, C): \langle F^k; C \rangle \rightarrow Sal(\mathcal{A})_{k-1}$ be the same characteristic map that identifies a $(k - 1)$ -cell $F^{k-1} \subset \partial F^k$ with the k -cell $\langle F^{k-1}; F^{k-1} \circ C \rangle \subset \partial \langle F^k; C \rangle$. Extend the map $\phi(F^k, C)$ to whole of $\langle F^k; C \rangle$ and use it as the attaching map, hence obtaining the k -skeleton. The boundary of every k -cell is given by

$$(2.1) \quad \partial \langle F^k; C \rangle = \bigcup_{F \prec F^k} \langle F; F \circ C \rangle.$$

Now we state the theorem that justifies the construction of this cell complex.

Theorem 2.11. *The regular CW complex $Sal(\mathcal{A})$ constructed above has the homotopy type of the tangent bundle complement $M(\mathcal{A})$.*

Proof. The above theorem is a special case of the theorem for submanifold arrangements. We refer the reader to [1, Theorem 3.3.7] for the details of the proof. \square

Example 2.12. As an example of this construction consider the arrangement of 2 points in a circle (Example 2.5). The Figure 3 below illustrates the dual cell structure induced by the arrangement and the associated Salvetti complex.

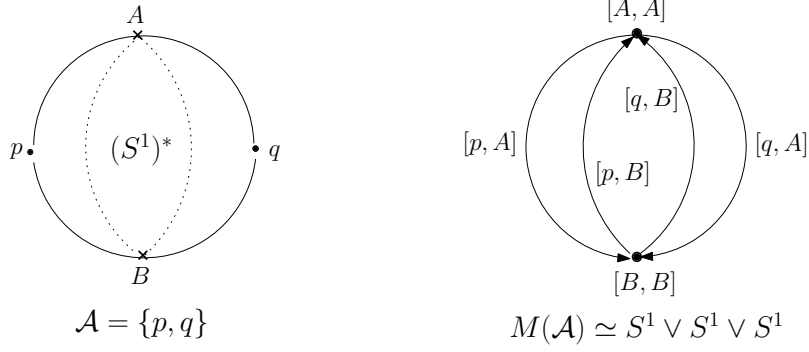


FIGURE 3. Arrangement in S^1 and the associated Salvetti complex

We now look at some obvious properties of the above defined CW structure and also infer some more information about the tangent bundle complement.

Theorem 2.13. *Let \mathcal{A} be an arrangement of hyperspheres in S^l and let $Sal(\mathcal{A})$ denote the associated Salvetti complex. Then*

- (1) *There is a natural cellular map $\psi: Sal(\mathcal{A}) \rightarrow \mathcal{F}^*(\mathcal{A})$ given by $[F^k, C] \mapsto F^k$. The restriction of ψ to the 0-skeleton is a bijection and in general*

$$\psi^{-1}(F^k) = \{C \in \mathcal{C}(\mathcal{A}) \mid C \prec F^k\}$$

- (2) *For every chamber C there is a cellular map $\iota_C: \mathcal{F}^*(\mathcal{A}) \rightarrow Sal(\mathcal{A})$ taking F^k to $[F^k, F^k \circ C]$ which is an embedding of $\mathcal{F}^*(\mathcal{A})$ into $Sal(\mathcal{A})$, and*

$$Sal(\mathcal{A}) = \bigcup_{C \in \mathcal{C}(\mathcal{A})} \iota_C(\mathcal{F}^*).$$

- (3) *The absolute value of the Euler characteristic of $M(\mathcal{A})$ is the number of chambers.*
 (4) *Let $T\mathcal{A}$ denote the union of the tangent bundles of the submanifolds in \mathcal{A} then,*

$$\text{rank } \tilde{H}^i(TS^l, T\mathcal{A}) = \begin{cases} |\chi(M(\mathcal{A}))| & \text{if } i = l \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Proofs of (1) and (2) are fairly straight forward. It follows that S^l is a retract of $M(\mathcal{A})$.

We prove (3) by explicitly counting cells in the Salvetti complex. The Euler characteristic of a CW complex K is equal to the alternating sum of number of cells of each dimension. Given a k -dimensional dual cell F^k there are as many as $|\{C \in \mathcal{C}(\mathcal{A}) \mid F \leq C\}|$ k -dimensional cells in $Sal(\mathcal{A})$. Hence for a vertex $[C, C] \in Sal(\mathcal{A})$ the number of k -dimensional cells that have this particular 0-cell as a vertex is equal to the number of k -faces of $\mathcal{F}^*(\mathcal{A})$ that contain C . The alternating sum of number of cells that contain a particular vertex C of $\mathcal{F}^*(\mathcal{A})$ is equal

to $1 - \chi(Lk(C))$, where $Lk(C)$ is the link of C in $X^*(\mathcal{A})$. Applying this we get,

$$\chi(Sal(\mathcal{A})) = \sum_{C \in \mathcal{C}(\mathcal{A})} (1 - \chi(Lk(C)))$$

Since S^l is compact all the chambers are bounded we have $Lk(C) \simeq S^{l-1}$. Thus,

$$\begin{aligned} \chi(Sal(\mathcal{A})) &= \sum_{C \in \mathcal{C}(\mathcal{A})} (1 - \chi(Lk(C))) \\ &= \sum_{C \in \mathcal{C}(\mathcal{A})} (1 - [1 + (-1)^{l-1}]) \\ &= (-1)^l \sum_{C \in \mathcal{C}(\mathcal{A})} 1 \end{aligned}$$

Hence,

$$\chi(M(\mathcal{A})) = (-1)^l (\text{number of chambers}).$$

Let $\bigcup \mathcal{A}$ denote the union of submanifolds in \mathcal{A} . Since \mathcal{A} induces a regular cell decomposition it has the homotopy type of wedge of $(l-1)$ -dimensional spheres. The number of spheres is equal to the number of chambers. The claim (4) follows from the homeomorphism of pairs $(TS^l, T\mathcal{A}) \cong (S^l, \bigcup \mathcal{A})$. \square

3. TOPOLOGY OF THE COMPLEMENT

Aim of this section is to investigate how does the combinatorics of the associated posets tell us about the topological invariants of the complement. We first derive a closed form formula for the homotopy type of the complement. Then we establish a connection between the intersection poset and the cohomology groups.

3.1. Closed form for the homotopy type. First we look at arrangements in S^1 . An arrangement in S^1 consists of n copies of S^0 , i.e. $2n$ points. The tangent bundle complement of such an arrangement is homeomorphic to the infinite cylinder with $2n$ punctures. Thus we have the following theorem.

Theorem 3.1. *Let \mathcal{A} be an arrangement of 0-spheres in S^1 . If $|\mathcal{A}| = n$ then*

$$M(\mathcal{A}) \simeq \bigvee_{2n+1} S^1.$$

From now on we assume that all our spheres are simply connected, we also restrict to centrally symmetric arrangements of spheres in order to avoid technicalities. We say that two arrangements are combinatorially isomorphic if their corresponding face posets and intersection posets are isomorphic.

Lemma 3.2. *Given a centrally symmetric sphere arrangement \mathcal{A} in S^l there exists a generic hypersphere S_0 such that S_0 intersects every member of \mathcal{A} in general position. Let S_0^+, S_0^- denote the connected components of $S^l \setminus S_0$ and $\mathcal{A}^+ := \mathcal{A}|S_0^+$, $\mathcal{A}^- := \mathcal{A}|S_0^-$ be the restricted arrangements. Then \mathcal{A}^+ and \mathcal{A}^- are combinatorially isomorphic arrangements of hyperplanes in S_0^+ and S_0^- (both $\cong \mathbb{R}^l$) respectively.*

Proof. Since the arrangement is centrally symmetric each individual hypersphere in \mathcal{A} is invariant under the antipodal mapping $x \mapsto -x$ of S^l . For every $S \in \mathcal{A}$ let $a(S) := S/(x \sim -x) \cong \mathbb{P}^{l-1}$, let S_0 be the equator with respect to this action and let S_0^+, S_0^- denote the hemispheres whose boundary is S_0 . This equator S_0 generically intersects with every S and $a(S) \setminus (S_0 \cap a(S))$ is a hyperplane contained in $S_0^+ \cong \mathbb{R}^l$. Under this correspondence an intersection of hyperspheres is mapped to the intersection of hyperplanes. The restrictions of the arrangement to the hemispheres S_0^+ and S_0^- gives us two hyperplane arrangements in \mathbb{R}^l which are combinatorially isomorphic. \square

Here are two well known facts that we need.

Lemma 3.3. *If (Y, A) is a CW pair such that the inclusion $A \hookrightarrow Y$ is null homotopic then $Y/A \simeq Y \vee SA$, where SA is the suspension of A .*

Proof. See [2, Chapter 0]. \square

Lemma 3.4. *Let \mathcal{B} be an essential and affine arrangement of hyperplanes in \mathbb{R}^l . Then the cell complex which is dual to the induced stratification is regular and homeomorphic to a closed ball of dimension l .*

Proof. See [4, Lemma 9]. \square

Now the main theorem of this section.

Theorem 3.5. *Let \mathcal{A} be a centrally symmetric arrangement of spheres in S^l . Let \mathcal{A}^+ and \mathcal{A}^- be the hyperplane arrangements as in Lemma 3.2 and $\mathcal{C}(\mathcal{A}^+)$ be the set of its chambers. Then the tangent bundle complement*

$$M(\mathcal{A}) \simeq \text{Sal}(\mathcal{A}^-) \vee \bigvee_{|\mathcal{C}(\mathcal{A}^+)|} S^l.$$

Proof. Let $C \in \mathcal{C}(\mathcal{A}^+)$ and let Q denote the dual cell complex $(S_0^+, \mathcal{F}^*(\mathcal{A}^+))$. Define the map ι_C^+ as follows:

$$\begin{aligned} \iota_C^+ : Q &\hookrightarrow \text{Sal}(\mathcal{A}) \\ F &\mapsto [F, F \circ C] \end{aligned}$$

Claim 1: The image of ι_C^+ , in $\text{Sal}(\mathcal{A})$, is homeomorphic to Q (which is a closed ball of dimension l).

Observe that ι_C^+ is just the restriction of the map ι_C , defined in Theorem 2.13, which is an embedding of S^l into $M(\mathcal{A})$. Hence ι_C^+ maps Q homeomorphically onto its image.

Hence ι_C^+ is the characteristic map which attaches the boundary ∂Q to the $(l-1)$ -skeleton of $\text{Sal}(\mathcal{A}^-)$. For notational simplicity let j_C denote the restriction of ι_C^+ to ∂Q .

Claim 2: The image of j_C is also the boundary of an l -cell in $\text{Sal}(\mathcal{A}^-)$.

Consider the subcomplex of $\text{Sal}(\mathcal{A}^-)$ given by the cells $\{[F, F \circ C] \mid F \in \mathcal{F}^*(\mathcal{A}^-)\}$. By Lemma 3.3 above this subcomplex is homeomorphic to the closed l -ball. The boundary of this closed ball is precisely the image of j_C .

Therefore the characteristic map ι_C^+ is the extension of j_C to the cone over ∂Q (which is Q). Hence j_C is null homotopic. Applying the above arguments to every chamber of \mathcal{A}^+ establishes the theorem. \square

We state the following obvious corollary for the sake of completeness.

Corollary 3.6. *Let \mathcal{A} be a centrally symmetric arrangement of spheres in S^l . With the notation as in Lemma 3.2 we have:*

$$\pi_1(M(\mathcal{A})) \cong \pi_1(M(\mathcal{A}^-)).$$

Example 3.7. Consider the arrangement of 2 circles in S^2 introduced in Example 2.6. It is clear that the arrangement \mathcal{A}^- in this case is the arrangement of two lines in \mathbb{R}^2 that intersect in a single point. Hence

$$M(\mathcal{A}) \simeq T^2 \vee S^2 \vee S^2 \vee S^2 \vee S^2.$$

The Salvetti complex consists of four 0-cells, eight 1-cells and eight 2-cells. The 2-torus T^2 in the above formula corresponds to $M(\mathcal{A}^-)$ and the 4 spheres correspond to chambers.

Example 3.8. Consider the arrangement of three circles in S^2 that intersect in general position. This arrangement arises as the intersection of S^2 with the coordinate hyperplanes in \mathbb{R}^3 . In this case \mathcal{A}^- is the arrangement of three lines in general position. Thus

$$M(\mathcal{A}) \simeq \text{Sal}(\mathcal{A}^+) \bigvee \vee_7 S^2.$$

Example 3.9. Finally, consider the arrangement of three S^2 s in S^3 that intersect like co-ordinate hyperplanes in \mathbb{R}^3 . The \mathcal{A}^- in this case is the arrangement of co-ordinate hyperplanes hence $\text{Sal}(\mathcal{A}^-) \simeq T^3$, the 3-torus. This arrangement has 8 chambers. So we have the following

$$M(\mathcal{A}) \simeq T^3 \bigvee \vee_8 S^2.$$

3.2. Cohomology of the Complement. We now establish a relationship between the cohomology of the tangent bundle complement and the intersection poset. Let \mathcal{A} be a centrally symmetric arrangement of spheres in S^l , let \mathcal{A}^+ be the affine hyperplane arrangement in the positive hemisphere. Let L and L^+ denote the corresponding intersection posets. Observe that the map from L to L^+ that sends $Y \in L$ to $Y|S_0^+ =: Y^+$ is one-to-one up to rank $l-1$. If L_{l-1} and L_{l-1}^+ denote the sub-posets consisting of elements of rank less than or equal to $l-1$ then the previous map is a poset isomorphism. For notational simplicity we use M^- for $M(\mathcal{A}^-)$.

Theorem 3.10. *With the notation above, we have the following*

$$\text{rank}(H^i(M, \mathbb{Z})) = \begin{cases} \sum_{\substack{Y \in L \\ \text{rk}(Y)=i}} |\mu(S^l, Y)| & \text{for } 0 \leq i < l \\ \sum_{Y \in L} |\mu(S^l, Y)| & \text{for } i = l \end{cases}$$

Where μ is the Möbius function of the intersection poset.

Proof. We use Theorem 3.5 above and [3, Proposition 3.75] in order to prove the assertion by considering two cases.

Case 1: $i < l$

$$\begin{aligned}
\text{rank}(H^i(M)) &= \text{rank}(H^i(M^-)) + \sum_{|\mathcal{C}(\mathcal{A}^+)|} \text{rank}(H^i(S^l)) \\
&= \text{rank}(H^i(M^-)) + 0 \\
&= \sum_{rk(Y^-)=i} \text{rank}(H^0(\coprod_{|\mu(Y^-)|} Y^-)) \\
&= \sum_{rk(Y)=i} \text{rank}(H^0(\coprod_{|\mu(Y)|} Y)) \\
&= \sum_{rk(Y)=i} |\mu(S^l, Y)|
\end{aligned}$$

The last equality follows from the fact that each Y is a sphere of dimension $l - i$.

Case 2: $i = l$

$$\begin{aligned}
\text{rank}(H^l(M)) &= \text{rank}(H^l(M^-)) + \sum_{|\mathcal{C}(\mathcal{A}^+)|} \text{rank}(H^l(S^l)) \\
&= \sum_{rk(Y^-)=l} |\mu(Y^-)| + |\mathcal{C}(\mathcal{A}^+)| \\
&= \sum_{rk(Y^-)=l} |\mu(Y^-)| + \sum_{Y^+ \in L^+} |\mu(Y^+)| \\
&= \sum_{Y \in L} |\mu(Y)|
\end{aligned}$$

The third equality follows from the expression for the number of chambers of an affine hyperplane arrangement. The last equality is true because the number of rank l elements in L are twice the corresponding number in L^- . \square

In particular the above theorem verifies [1, Conjecture 3.7.8] for sphere arrangements. The cohomology ring of the tangent bundle complement in this case can be expressed as a direct sum of an Orlik-Solomon algebra and some top dimensional classes. The number of these top dimensional classes is equal to the number of graded pieces in the Orlik-Solomon algebra. If \mathcal{A} is a centrally symmetric arrangement of spheres then one might call the cohomology algebra $H^*(M(\mathcal{A}), \mathbb{Z})$ the *spherical Orlik-Solomon algebra*.

4. THE FUNDAMENTAL GROUP

A path in the (regular) cell complex is a sequence of consecutive edges and its length is the number of edges. A minimal path is path of shortest length among all the paths that join its end points. In case of a an oriented 1-skeleton by a positive path we mean a path all of whose edges have same direction.

Lemma 4.1. *Let \mathcal{A} be an arrangement of spheres in $S^l, l \geq 2$ then any two positive minimal paths in the 1-skeleton of $\text{Sal}(\mathcal{A})$ that have same initial as well as terminal vertex are homotopic relative to $\{0, 1\}$.*

Proof. Given two positive minimal paths α, β in $Sal(\mathcal{A})$ with the same end points apply the retraction map to get paths in \mathcal{F}^* . Observe that no two edges of these two paths are sent to a same edge of \mathcal{F}^* . The conclusion follows from the fact that \mathcal{F}^* is simply connected. See also Deshpande [1, Theorem 3.5.5], Salvetti [5, Theorem 17] for a proof. \square

Given an arrangement \mathcal{A} let \mathcal{G}^+ denote the associated *positive category*. It is defined to be the category of directed paths in the Salvetti complex $Sal(\mathcal{A})$. The objects of this category are the vertices of the Salvetti complex and morphisms are directed homotopy classes of positive paths (i.e. two such paths are connected by a sequence of substitutions of minimal positive paths). For a path α its equivalence class in \mathcal{G}^+ is denoted by $[\alpha]_+$. Let \mathcal{G} denote the arrangement groupoid. It is basically the fundamental groupoid of the associated Salvetti complex. For a path α its equivalence class in \mathcal{G} is denoted by $[\alpha]$. Since \mathcal{G} is the category of fractions of \mathcal{G}^+ we denote by $J: \mathcal{G}^+ \rightarrow \mathcal{G}$ the associated canonical functor.

Theorem 4.2. *Let \mathcal{A} be a centrally symmetric arrangement of spheres in $S^l (l \geq 2)$ then the associated canonical functor $J: \mathcal{G}^+ \rightarrow \mathcal{G}$ is faithful on the class of minimal positive paths.*

Proof. We already know that if α, β are two minimal positive paths with the same end points then $[\alpha] = [\beta]$ in $\mathcal{G}(\mathcal{A})$. Hence it is enough to show that $[\alpha]_+ = [\beta]_+$. We argue on the lines of the proof of [5, Theorem 20]. Since each $S \in \mathcal{A}$ is centrally symmetric around the origin the antipodal map induces a fixed point free cellular action on the faces of \mathcal{A} .

Suppose $\alpha = (a_1, \dots, a_n)$ and $\beta = (b_1, \dots, b_n)$ are two minimal positive paths in $G(\mathcal{A})$ that start at C and end at D . We proceed by induction on n , cases $n = 0, 1$ being trivial. Assume that the statement is true for all minimal positive paths with same end points and of length strictly less than n . If $a_1 = b_1$ then we are done by induction. Hence assume that a_1, b_1 are distinct and are dual to the hyperspheres S_a, S_b respectively.

We have that $S_a, S_b \in R(C, D)$ (the set of hyperspheres separating C and D) and $S_a \cap S_b \cong S^{l-2}$, since both these hyperspheres are equatorial. For every $S \in \mathcal{A}$ let $X_S(C, D)$ denote the closures of the connected components of $S^l \setminus S$ that contain either C or D (or both) and let $\mathcal{H}(C, D) = \cap_{S \in \mathcal{A}} X_S(C, D)$. Note that if $S \in R(C, D)$ then $X_S(C, D)$ contains closures of both the components of $S^l \setminus S$. On the other hand if $S \notin R(C, D)$ then $X_S(C, D)$ is homeomorphic to the l -disk \mathbb{D}^l .

Claim 1: The set $\mathcal{H}(C, D)$ is either connected or empty.

$$\begin{aligned} \mathcal{H}(C, D) &= \cap_{S \in \mathcal{A}} X_S(C, D) \\ &= (\cap_{S \in R(C, D)} X_S(C, D)) \cap (\cap_{S \notin R(C, D)} X_S(C, D)) \\ &= (\cap_{S \in R(C, D)} S) \cap (\cap_{S \notin R(C, D)} \mathbb{D}^l) \end{aligned}$$

The definition of sphere arrangements implies that the first intersection is either a sphere of some finite dimension or it is empty. Hence the set $\mathcal{H}(C, D)$ is either homeomorphic to some closed disk (of possibly lower dimension) or it is empty.

Claim 2: If $\mathcal{H}(C, D) \neq \emptyset$ then $\mathcal{H}(C, D) \cap S_a \cap S_b \neq \emptyset$.

Let S_a^+, S_a^- (respectively S_b^+, S_b^-) denote the (closures of the) connected components of $S_a \setminus (S_a \cap S_b)$ (respectively $S_b \setminus (S_a \cap S_b)$). Without loss of generality assume that S_a^+ and S_b^+ intersect \overline{C} . This implies

$$\mathcal{H}(C, D) \cap S_a^+ \neq \emptyset \neq \mathcal{H}(C, D) \cap S_b^+.$$

A similar argument using \overline{D} establishes the claim.

Hence the set $\mathcal{H}(C, D) \cap S_a \cap S_b$ contains a codimension 2 face say F^2 . Let C' denote $F^2 * C$. Since \mathcal{A} is an arrangement of submanifolds there exists a minimal positive path γ_0 from C' to D . Also, there exist two minimal positive paths γ_1, γ_2 such that γ_1 starts at $a_1 \circ C$ and γ_2 starts at $b_1 \circ C$ such that both of them end at C' . Using this we can construct two new minimal positive paths $\eta = a_1 \gamma_1 \gamma_0$ and $\eta' = b_1 \gamma_2 \gamma_0$. The paths α, η are minimal positive with the same end points and share the same first edge. Hence by induction, $[\alpha]_+ = [\eta]_+$. For the same reasons $[\beta]_+ = [\eta']_+$. If $C' \neq D$ then the path γ_0 is of nonzero length and again by induction $[a_1 \gamma_1]_+ = [b_1 \gamma_2]_+$ implying $[\eta]_+ = [\eta']_+$. Now the transitivity of the equivalence relation establishes the claim.

The cases in which either $C' = D$ or $\mathcal{H}(C, D) = \emptyset$ can be treated similarly. \square

We call such an arrangement a *flat arrangement*. For a flat arrangement \mathcal{A} , we indicate by $[\mu(C \rightarrow D)]$ the unique equivalence class (in $\mathcal{G}^+(\mathcal{A})$ or $\mathcal{G}(\mathcal{A})$) determined by a minimal positive path from a chamber C to another chamber D .

Definition 4.3. An arrangement of hyperspheres \mathcal{A} in a S^l is said to have the *involution property* if and only if there exists a graph automorphism $\phi: \mathcal{F}_1^* \rightarrow \mathcal{F}_1^*$ of the dual 1-skeleton (considered as a graph) satisfying:

- (1) ϕ is an involution (which induces involution on the vertices as well as the edges);
- (2) for every vertex C , $d(C, \phi(C)) = \max_{D \in \mathcal{F}_0^*} d(C, D)$;
- (3) $d(C, \phi(C)) = d(C, D) + d(D, \phi(C))$ for every vertex C and D .

Lemma 4.4. Let \mathcal{A} be a centrally symmetric sphere arrangement in S^l then \mathcal{A} has the involution property.

Proof. The antipodal action on S^l provides the required graph automorphism on the 1-skeleton of the associated Salvetti complex. \square

The image of either a vertex or an edge under ϕ will be denoted by writing $\#$ on its top, for example, $C^\# := \phi(C)$.

Lemma 4.5. If \mathcal{A} is a hypersphere arrangement with the involution property then:

- (1) $d(C, C^\#) = |\mathcal{A}|$ (number of spheres in \mathcal{A}) for all C .
- (2) $d(C, D) = d(C^\#, D^\#)$ for all C, D .

Proof. Using the property (3) in Definition 4.3 we have:

$$(4.1) \quad d(C, C^\#) = d(C, D) + d(D, C^\#)$$

$$(4.2) \quad d(C, C^\#) = d(C, D^\#) + d(D^\#, C^\#)$$

Adding equations (4.1) and (4.2) we get

$$2d(C, C^\#) = 2d(D, D^\#)$$

Without loss of generality assume that $d(C, C^\#) = |\mathcal{A}| - 1$. Hence there is a hypersphere $S \in \mathcal{A}$ such that $S \notin R(C, C^\#)$. Choose C' such that $S \in R(C, C')$. Which implies $d(C, C') > d(C, C^\#)$, a contradiction. Therefore no such S exists, which proves (1). We call the number $d(C, C^\#) = |\mathcal{A}|$, the *diameter* of S^l (with respect to \mathcal{A}).

Now subtracting $d(D, D^\#) = d(D, C^\#) + d(C^\#, D^\#)$ from (4.1) we get

$$0 = d(C, D) - d(C^\#, D^\#)$$

which proves (2). \square

This involution also preserves the positive equivalence on paths as proved in the next lemma.

Lemma 4.6. *If \mathcal{A} is a hypersphere arrangement with the involution property then the involution ϕ induces a functor on \mathcal{G}^+ which is also an involution.*

Proof. We start by showing that there is a bijection between the set of edge-paths of \mathcal{F}_1^* and the set of all positive paths in $Sal(\mathcal{A})_1$. In particular this bijection is given by $[F^1, C] \mapsto F^1$. Extend the given involution to $Sal(\mathcal{A})_1$ by sending $[F^1, C]$ to $[(F^1)^\#, C^\#]$. Under this involution a positive path $\alpha = (a_1, \dots, a_n)$ goes to a positive path

$$\alpha^\# := (a_1^\#, \dots, a_n^\#)$$

If γ_1, γ_2 are two minimal positive boundary paths of a 2-cell in $Sal(\mathcal{A})$ then so are $\gamma_1^\#, \gamma_2^\#$. Therefore $[\gamma_1]_+ = [\gamma_2]_+ \Rightarrow [\gamma_1^\#]_+ = [\gamma_2^\#]_+$. \square

As the functor J is faithful on the class of minimal positive paths there is only one positive equivalence class of such paths. We denote this unique equivalence class by the symbol $[\mu(C \rightarrow C^\#)]$. Further define a ‘positive’ loop based at $[C, C]$ as

$$\delta(C) := \mu(C \rightarrow C^\#)\mu(C^\# \rightarrow C)$$

By $\delta^k(C)$ we mean that this particular loop is traversed k times in the same direction if $k > 0$ and in the reverse direction if $k < 0$. We will say that a positive path α begins (or ends) with a positive path α' if and only if $\alpha = \alpha'\beta (= \beta\alpha')$ for some positive path β .

Lemma 4.7. *Let \mathcal{A} be a hypersphere arrangement with the involution property and α be a positive path from C to D . Then:*

- (1) $[\alpha][\mu(D \rightarrow D^\#)] = [\mu(C \rightarrow C^\#)][\alpha^\#]$;
- (2) *if for a chamber D' , β is some positive path from C to D' then $\alpha\delta^n(D)$ begins with β ;*
- (3) *if $[\gamma] \in \mathcal{G}(C, D)$ then there exists $n \in \mathbb{N}$ and a positive path γ' such that*

$$[\gamma] = [\delta^{-n}(C)][\gamma'].$$

Proof. For (1) we use induction on the length of α . In fact, it is enough to assume that $\alpha = \mu(C \rightarrow C_1)$ such that $d(C, C_1) = 1$. Thus:

$$\begin{aligned} \alpha\mu(C_1 \rightarrow C_1^\#) &= \mu(C \rightarrow C_1)\mu(C_1 \rightarrow C_1^\#) \\ &\stackrel{+}{\sim} \mu(C \rightarrow C_1)\mu(C_1 \rightarrow C^\#)\mu(C^\# \rightarrow C_1^\#) \\ &\stackrel{+}{\sim} \mu(C \rightarrow C^\#)\mu(C^\# \rightarrow C_1^\#) \\ &\stackrel{+}{\sim} \mu(C \rightarrow C^\#)\alpha^\# \end{aligned}$$

By the same arguments, the following stronger statement is true:

$$(4.3) \quad [\alpha][\delta^k(D)] = [\delta^k(C)][\alpha], \quad k \geq 1$$

For (2), let $\beta = (b_1, \dots, b_n)$ where b_i is an edge from B_{i-1} to B_i ($B_0 = C, B_n = D'$). Observe that $\beta\mu(B_n \rightarrow B_{n-1}^\#) = (b_1, \dots, b_{n-1})\mu(B_{n-1} \rightarrow B_{n-1}^\#)$. By induction on n assume that there exists a positive path η from B_{n-1} to D such that -

$$(b_1, \dots, b_{n-1})\eta = \alpha\delta^{n-1}(D).$$

Using (1), we get

$$\beta\mu(B_n \rightarrow B_{n-1}^\#)\eta^\# = (b_1, \dots, b_{n-1})\eta\delta(D) = \alpha\delta^n(D)$$

which proves (2).

For γ an arbitrary path from C to D assume $\gamma = (\epsilon_1 a_1, \dots, \epsilon_n a_n), \epsilon_i \in \{\pm 1\}$. Let $A_i = t(\epsilon_1 a_1, \dots, \epsilon_i a_i)$. Set $k = |\{1 \leq i \leq n \mid \epsilon_i = -1\}|$, we prove (3) by induction on k . The case $k = 0$ is clear since it means that γ is a positive path. Assume that the statement is true for $k - 1$. Now the general case; there exists an index j such that $\epsilon_1 = \dots = \epsilon_{j-1} = 1$ and $\epsilon_j = -1$. We have

$$\begin{aligned} \delta(C)\gamma &= \mu(C \rightarrow C^\#)\mu(C^\# \rightarrow C)(a_1, \dots, a_{j-1}, -a_j, \epsilon_{j+1}a_{j+1}, \dots, \epsilon_n a_n) \\ &\stackrel{\pm}{\sim} \mu(C \rightarrow C^\#)a_1^\# \mu(A_1^\# \rightarrow A_1)(a_2, \dots, a_{j-1}, -a_j, \epsilon_{j+1}a_{j+1}, \dots, \epsilon_n a_n) \text{ (from 1)} \\ &\stackrel{\pm}{\sim} \mu(C \rightarrow C^\#)a_1^\# \dots a_{j-1}^\# \mu(A_{j-1}^\# \rightarrow A_{j-1})(-a_j, \epsilon_{j+1}a_{j+1}, \dots, \epsilon_n a_n) \\ &\stackrel{\pm}{\sim} \mu(C \rightarrow C^\#)a_1^\# \dots a_{j-1}^\# \mu(A_{j-1}^\# \rightarrow A_j)(\epsilon_{j+1}a_{j+1}, \dots, \epsilon_n a_n) \\ &\stackrel{\pm}{\sim} \delta^{1-n}(C)\gamma' \text{ (by induction hypothesis,)} \end{aligned}$$

where γ' is a positive path. Hence $[\gamma] = [\delta^{-n}(C)]\gamma'$. \square

Recall that [6, Section 0.5.7] the *word problem* for a group G is the problem of deciding whether or not an arbitrary word w in G is the identity of G . The word problem for G is *solvable* if and only if there exists an algorithm to determine whether $w = 1_G$ or equivalently, if there exists an algorithm to determine when two arbitrary words represent the same element of G .

Theorem 4.8. *Let \mathcal{A} be a hypersphere arrangement with the involution property. Then if $J: \mathcal{G}^+(\mathcal{A}) \rightarrow \mathcal{G}(\mathcal{A})$ is faithful the word problem for $\pi_1(M(\mathcal{A}))$ is solvable.*

Proof. Let $[\alpha], [\beta]$ be two loops in $\pi_1(\text{Sal}(\mathcal{A}))$ based at a vertex $[C, C]$. Then according to Lemma 4.7 there is a finite algorithm to write -

$$[\beta] = [\delta^{-k}(C)][\beta'], \quad [\alpha] = [\delta^{-k}(C)][\alpha']$$

where β', α' are positive loops based at $[C, C]$. Hence, $[\alpha] = [\beta]$ if and only if $[\alpha']_+ = [\beta']_+$. The theorem follows because there are only finitely many positive paths of given length to choose from. \square

Lemma 4.9. *Let \mathcal{A} be a hypersphere arrangement of $S^l, l \geq 2$ then $\pi_n(M(\mathcal{A})) \neq 0$ for $n \geq 2$.*

Proof. It follows from Theorem 2.13 that the map $\psi_*: \pi_n(M(\mathcal{A})) \rightarrow \pi_n(S^l)$ is surjective. \square

5. ARRANGEMENTS OF PROJECTIVE SPACES

Next we consider the arrangements in projective spaces. Given a finite dimensional real projective space \mathbb{P}^l we consider a finite collection of subspaces that are homeomorphic to \mathbb{P}^{l-1} . We define the projective arrangements as follows.

Definition 5.1. Let \mathbb{P}^l denote the l -dimensional projective space and $a: S^l \rightarrow \mathbb{P}^l$ be the antipodal map. A finite collection $\mathcal{A} = \{H_1, \dots, H_n\}$ of codimension 1 projective spaces is called an *arrangement of projective spaces* (or a *projective arrangement*) if and only if $\tilde{\mathcal{A}} = \{a^{-1}(H) \mid H \in \mathcal{A}\}$ is a centrally symmetric arrangement of spheres in S^l .

It is not hard to see that the above defined arrangements are indeed arrangements of submanifolds. The homotopy type of the tangent bundle complement associated to a projective arrangement is easier to understand because of the antipodal action.

Theorem 5.2. *Let \mathcal{A} be a projective arrangement in \mathbb{P}^l and $\tilde{\mathcal{A}}$ be the corresponding centrally symmetric sphere arrangement in S^l . Then the antipodal map on the sphere extends to its tangent bundle and*

$$M(\mathcal{A}) \cong M(\tilde{\mathcal{A}})/((x, v) \sim a(x, v)).$$

Proof. If (x, v) is a point in the tangent bundle of S^l extend the antipodal map in the obvious way, $a(x, v) = (-x, v)$. We now prove that the space $M(\tilde{\mathcal{A}})$ is a covering space of $M(\mathcal{A})$. This follows from the fact that $a: TS^l \rightarrow T\mathbb{P}^l$ is a covering map for every l . Note that the antipodal map is also cellular on the faces of the arrangement.

Consequently it induces a cellular map on $Sal(\tilde{\mathcal{A}})$ by sending a cell $[F, C]$ to $[a(F), a(C)]$. Hence we get a cell structure for $Sal(\mathcal{A})$. In particular the tangent bundle complement associated to a projective arrangement contains a wedge of projective spaces. Hence there is a torsion in the homology as well as the fundamental group of the tangent bundle complement. Moreover $\pi_1(M(\tilde{\mathcal{A}}))$ is an index 2 subgroup of $\pi_1(M(\mathcal{A}))$. \square

Example 5.3. Consider a projective arrangement \mathcal{A} in \mathbb{P}^2 corresponding to the arrangement of 2 circles in S^2 (Example 2.6). In this projective arrangement we have two \mathbb{P}^1 's intersecting in a point and there are two chambers. Taking the quotient as above of the space obtained in Example 3.7 we get the following

$$M(\mathcal{A}) \simeq K \vee \mathbb{P}^2 \vee \mathbb{P}^2$$

where K denotes the Klein bottle. Recall that the 2-torus is a two-fold cover of the Klein bottle.

Given a projective arrangement \mathcal{A} let $J: \mathcal{G}^+ \rightarrow \mathcal{G}$ denote the canonical functor between the positive category and the arrangement groupoid. For the corresponding (centrally symmetric) sphere arrangement $\tilde{\mathcal{A}}$ let $\tilde{J}: \tilde{\mathcal{G}}^+ \rightarrow \tilde{\mathcal{G}}$ be the associated canonical functor. Because of the antipodal action on S^n the arrangement $\tilde{\mathcal{A}}$ has the involution property (Definition 4.3). It follows from Lemma 4.6 that this action induces an ‘antipodal’ functor on \mathcal{G}^+ . Recall that under this functor an object C (which is a chamber) is mapped to its antipodal (chamber) $C^\#$ and a morphism $[\alpha]$ is mapped to $[\alpha^\#]$.

Lemma 5.4. *With the notation as above the following diagram commutes:*

$$\begin{array}{ccc} \tilde{\mathcal{G}}^+ & \xrightarrow{\tilde{J}} & \tilde{\mathcal{G}} \\ \Phi^+ \downarrow & & \downarrow \Phi \\ \mathcal{G}^+ & \xrightarrow{J} & \mathcal{G} \end{array}$$

where the functor Φ^+ identifies antipodal objects and morphisms and Φ is the covering functor.

Proof. Follows from a simple diagram chase and the fact that S^l is the universal cover of $\mathbb{R}P^l$. \square

An immediate consequence of the lemma is -

Corollary 5.5. *The restriction of J to the class of minimal positive paths is faithful and the word problem for $\pi_1(M(\mathcal{A}))$ is solvable. Moreover if \mathcal{A} is a simplicial arrangement then J is faithful.*

Proof. The first statement follows from the commutativity of the diagram in the previous lemma. If $[\alpha]_+$ is a class of minimal positive path in \mathcal{G}^+ then the class representing either of α 's lift is also minimal positive in $\tilde{\mathcal{G}}^+$. If there are two distinct classes of minimal positive paths then first applying \tilde{J} to their lifts in $\tilde{\mathcal{G}}^+$ and then applying Φ results in producing two distinct classes of minimal positive paths in \mathcal{G} . By the same argument if \tilde{J} is faithful then J is also faithful.

Let us see why the word problem is solvable. Let $[\alpha]$ be a loop based at a vertex C in \mathcal{G} . Let $[\tilde{\alpha}]$ be the class representing a loop based at \tilde{C} (a vertex in the fiber over C). Then by statement 3 in Lemma 4.7 we have the following

$$[\tilde{\alpha}] = [\delta^{-n}(\tilde{C})][\tilde{\alpha}']$$

where $\delta^{-n}(\tilde{C}) = \mu(\tilde{C} \rightarrow \tilde{C}^\#)\mu(\tilde{C}^\# \rightarrow \tilde{C})$ and $\tilde{\alpha}'$ is a positive loop based at \tilde{C} . Since Φ is the covering functor, $\Phi([\delta^{-n}(\tilde{C})]) = [\delta^{-2n}(C)]$ here $\delta(C)$ is a positive loop based at C which traverses every vertex twice. Let $[\alpha']$ be the image $\Phi([\tilde{\alpha}'])$, it represents a class positive loop based at C . Note that choosing another lift of α that is based at the antipodal point $\tilde{C}^\#$ does not make any difference. Hence we have proved that any loop in $Sal(\mathcal{A})$ can be expressed as a composition of a ‘special loop’ (which traverses each vertex a fixed number of times) and a positive loop. Now the same argument as in the proof of Theorem 4.8 shows that the word problem for $\pi_1(M(\mathcal{A}))$ is solvable. \square

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